

Boltzmann Enhancements of Biquasile Counting Invariants

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Abstract

In this paper, we build on the biquasiles and dual graph diagrams introduced in [7]. We introduce *biquasile Boltzmann weights* that enhance the previous knot coloring invariant defined in terms of finite biquasiles and provide examples differentiating links with the same counting invariant, demonstrating that the enhancement is proper. We identify conditions for a linear function $\phi : \mathbb{Z}_n[X^3] \rightarrow \mathbb{Z}_n$ to be a Boltzmann weight for an Alexander biquasile X .

KEYWORDS: biquasiles, dual graph diagrams, enhancements of counting invariants, Boltzmann weights

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1 Introduction

In [7], the second and third listed authors introduced a combinatorial structure known as *dual graph diagrams* for representing oriented knots and links and a corresponding algebraic structure known as *biquasiles* for defining knot and link invariants via counting vertex colorings of dual graph diagrams with biquasile elements satisfying certain conditions. Dual graph diagrams arise from taking both checkerboard graphs of a knot or link diagram together, sometimes called the *overlaid Tait graph*, and adding edge decorations to indicate crossing signs and orientations. Biquasiles are algebraic structures consisting of two quasigroup operations on a set X which interact according to certain identities, analogous in some sense to the two group structures on a field interacting via the distributive law.

Given an oriented knot or link L and a finite biquasile X , the number of vertex colorings of the corresponding dual graph diagram by X is unchanged by Reidemeister moves and hence defines an integer valued computable invariant of oriented knots and links. Starting with [3] and continuing with subsequent papers such as [2, 4, 6] and many more, counting invariants of knotted objects associated to various algebraic structures such as quandles, biquandles, racks, biracks, kei and bikei have been enhanced to obtain stronger invariants by defining invariants ϕ of colored knots or links. The resulting multiset of ϕ -values over the set of colorings of a knot or link then defines a generally stronger invariant whose cardinality recovers the original counting invariant.

In this paper we enhance the biquasile counting invariant with *Boltzmann weights*, functions from the set of ordered triples of elements of a biquasile X to an abelian group A with the property that the sum of Boltzmann weights at crossings is unchanged by Reidemeister moves and hence defines an A -valued invariant of X -colored dual graph diagrams under Reidemeister equivalence, analogously to the quandle and biquandle 2-cocycle invariants studied in [3, 5] etc. The paper is organized as follows. In Section 2 we recall the basics of dual graph diagrams and biquasiles. In Section 3 we define Boltzmann enhancements and provide examples. We close in Section 4 with some questions for future research.

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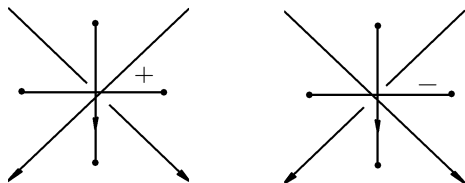
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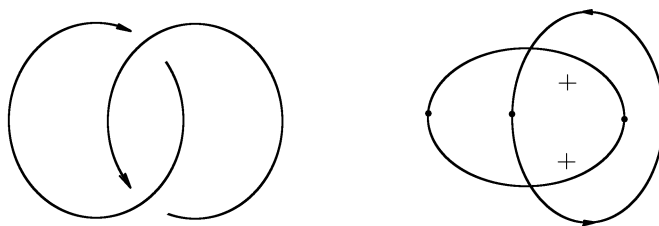
2 Dual Graph Diagrams and Biquasiles

We begin with two notions from [7].

Definition 1. Let D be an oriented knot or link diagram. We form the *dual graph diagram* associated to D by placing a vertex in every region and making two regions adjacent if they are opposite at a crossing. We then give each edge either a direction or a $+$ or $-$ sign as depicted:

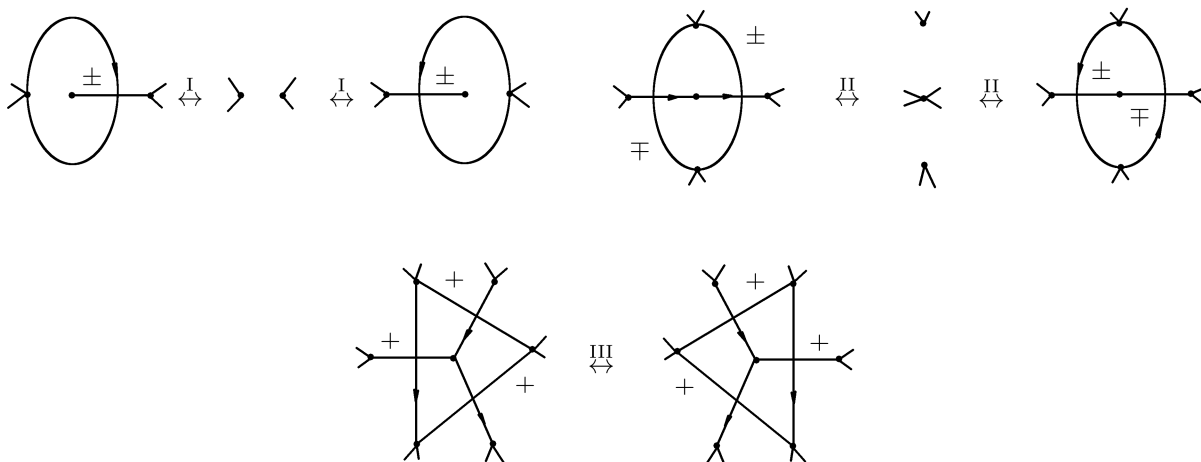


Example 1. The Hopf link below has the pictured dual graph diagram.



As noted in [7], dual graph diagrams are always pairs of dual planar graphs with the property that each edge crosses exactly one other edges, with crossing edges forming pairs where one edge has a sign and the other has a direction.

Definition 2. Two dual graph diagrams are *equivalent* if they are related by a sequence of the following moves:



These dual graph moves form a generating set of the oriented Reidemeister move expressed in a dual graph format. Next, we define biquasiles.

Definition 3. Let X be a set with binary operations $*, \cdot, \setminus^*, /^*, \setminus, / : X \times X \rightarrow X$ satisfying

$$\begin{aligned} y \setminus^* (y * x) &= x = (x * y) /^* y \\ y \setminus (y \cdot x) &= x = (x \cdot y) / y. \end{aligned}$$

Then we say X is a *biquasile* if for all $a, b, x, y \in X$ we have

$$\begin{aligned} a * (x \cdot [y * (a \cdot b)]) &= (a * [x \cdot y]) * (x \cdot [y * ([a * (x \cdot y)] \cdot b)]) & (i) \\ y * ([a * (x \cdot y)] \cdot b) &= (y * [a \cdot b]) * ([a * (x \cdot [y * (a \cdot b)])] \cdot b) & (ii). \end{aligned}$$

Example 2. Let R be any commutative ring with identity and let $d, s, n \in R$ be units. Then X is a biquasile under the operations

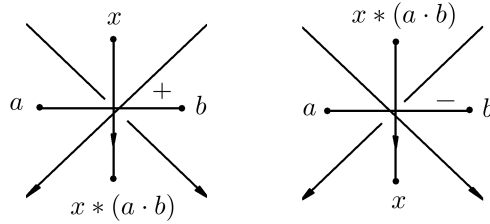
$$x \cdot y = dx + sy \quad \text{and} \quad x * y = -dsn^2x + ny.$$

Such a biquasile is called an *Alexander biquasile*; see [7] for more.

Example 3. For any finite set $X = \{x_1, \dots, x_n\}$ we can define a biquasile structure on X by choosing operation tables for $*$ and \cdot so that the biquasile axioms are satisfied. To save writing, we can drop the “ x ”s and write only the subscripts, resulting in a block matrix. For example, the Alexander biquasile structure on $\mathbb{Z}_3 = \{x_1 = 1, x_2 = 2, x_3 = 3\}$ (where we use 3 for the class of zero in \mathbb{Z}_3 since we start numbering our rows and columns with 1 instead of 0) with $d = 1, s = 1$ and $n = 2$ has operations $x * y = -dsn^2 + ny = 2x + 2y$ and $x \cdot y = dx + sy = x + y$ with operation tables and matrix

$$\begin{array}{c|ccc} * & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{array} \quad \begin{array}{c|ccc} \cdot & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 \end{array} \quad \leftrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 & 1 & 2 \\ 2 & 1 & 3 & 1 & 2 & 3 \end{array} \right].$$

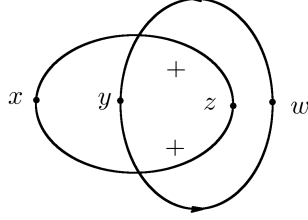
Definition 4. Given a dual graph diagram D and a biquasile X , an X -*coloring* of D is an assignment of elements of X to the vertices of D such that at every crossing we have the following pictures:



The biquasile axioms are chosen so that given a valid X -coloring of a diagram before a move, there is a unique corresponding valid coloring of the resulting diagram after the move. It follows that the number of X -colorings of a dual graph diagram is an invariant of oriented knots and links, denoted $\Phi_X^{\mathbb{Z}}(L)$, called the *biquasile counting invariant*.

Definition 5. Let L be a dual graph diagram and X the set of its vertices. Then the *fundamental biquasile* of L is the biquasile with presentation $\langle X \mid R \rangle$ where for each edge crossing, a relation is defined by the figure above. More precisely, the elements of the fundamental biquasile are equivalence classes of biquasile words in generators corresponding to the vertices of the dual graph diagram modulo the equivalence relation generated by the crossing relations and biquasile axioms. See [7] for more.

Example 4. Let us assign generators to the vertices in the Hopf link dual graph as pictured.



Reading the crossing relations from the diagram, the Hopf link has fundamental biquasile presentation

$$\mathcal{FB}(L) = \langle x, y, z, w \mid y = w * (x \cdot z), w = y * (x \cdot z) \rangle.$$

Then with coloring biquasile X given by the Alexander biquasile \mathbb{Z}_3 with $d = s = 1$ and $n = 2$, we obtain coloring equations

$$\begin{aligned} y &= w * (x \cdot z) \\ &= x + z + 2w \quad \text{and} \\ w &= y * (x \cdot z) \\ &= x + z + 2y \end{aligned}$$

so we have a homogeneous system of linear equations over \mathbb{Z}_3 with coefficient matrix

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}.$$

After row reduction over \mathbb{Z}_3 , we obtain

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the kernel has dimension 3 and we have $\Phi_X^{\mathbb{Z}_3}(L) = 3^3 = 27$.

3 Boltzmann Enhancements

In this section we will enhance the biquasile counting invariant using Boltzmann weights valued in an abelian group A , a strategy which has proved effective in the cases of other knot coloring structures such as quandles and biquandles.

Definition 6. Let X be a biquasile and A an abelian group. Then a *biquasile Boltzmann weight* is an A -linear map $\phi : A[X^3] \rightarrow A$ such that for all $x, y, a, b \in X$ we have

(i)

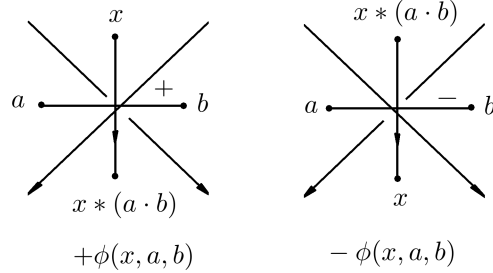
$$\phi(x, a, a \setminus (x \setminus^* x)) = \phi(x, (x \setminus^* x) / b, b) = 0$$

and

(ii)

$$\begin{aligned} &\phi(x, a, b) + \phi(b, x * (a \cdot b), y) + \phi(x * (a \cdot b), a, b * ([x * (a \cdot b)] \cdot y)) \\ &= \phi(b, x, y) + \phi(x, a, b * (x \cdot y)) + \phi(b * (x \cdot y), x * (a \cdot [b * (x \cdot y)]), y). \end{aligned}$$

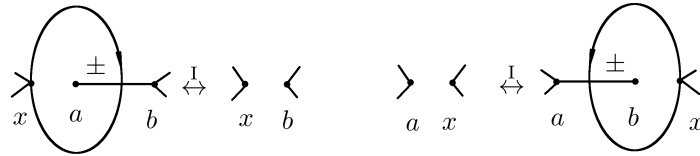
The biquasile Boltzmann weight axioms are chosen so that the sum of Boltzmann weight values according to the rules



are unchanged by dual graph Reidemeister moves as expressed in Definition 2. We prove this in the following proposition:

Proposition 1. *Let D be a dual graph diagram with a coloring by a biquasile X . Then if $\phi : X^3 \rightarrow A$ is a biquasile Boltzmann weight, the sum of the ϕ values over all edge crossings in D is unchanged by dual graph Reidemeister moves.*

Proof. Comparing the two sides of the dual graph Reidemeister I moves as labeled,



we have the requirement that

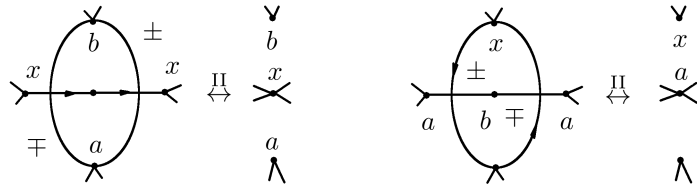
$$\pm\phi(x, a, b) = 0$$

when $x = x * (a \cdot b)$. Solving for a and b , this yields the requirements that

$$\pm\phi(x, a, a \setminus (x \setminus * x)) = \pm\phi(x, (x \setminus * x) / b, b) = 0$$

for all $x, a, b \in X$.

Our choice of $\phi(x, a, b)$ at positive crossings and $-\phi(x, a, b)$ at negative crossings (with x the output label) satisfies the Reidemeister II moves. More precisely, comparing the two sides of the dual graph Reidemeister II moves as labeled yields $\phi(x, a, b) - \phi(x, a, b)$ and 0 respectively:



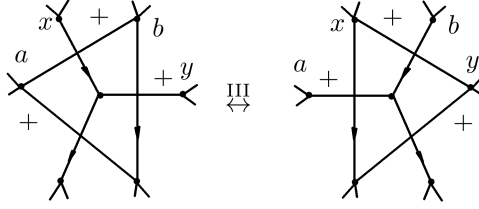
Finally, comparing the two sides of the dual graph Reidemeister III move as labeled yields

$$\phi(x, a, b) + \phi(b, x * (a \cdot b), y) + \phi(x * (a \cdot b), a, b * ([x * (a \cdot b)] \cdot y))$$

on one side and

$$\phi(b, x, y) + \phi(x, a, b * (x \cdot y)) + \phi(b * (x \cdot y), x * (a \cdot [b * (x \cdot y)]), y)$$

on the other:



□

Corollary 2. *If X is a biquasile and $\phi : A[X^3] \rightarrow A$ is a biquasile Boltzmann weight, then the multiset $\Phi_X^{M,\phi}(L)$ of ϕ values over the set of X -colorings of a dual graph diagram D representing an oriented knot or link L is an invariant of knots and links.*

We can convert the multiset of Boltzmann weights into a “polynomial” form by converting multiplicities to coefficients and elements to exponents of a formal variable u for ease of comparison, e.g. the multiset $\{0, 0, 0, 1, 1, 2, 3, 3\}$ becomes $3 + 2u + u^2 + 2u^3$. With this notation, we will write

$$\Phi_X^\phi(L) = \sum_{f \in \{X\text{-colorings}\}} u^{BW(f)}$$

and call $\Phi_X^\phi(L)$ the *Boltzmann enhanced polynomial* of L with respect to the biquasile X and the Boltzmann weight ϕ . While this convention only yields a true (Laurent) polynomial if the abelian group A is the integers \mathbb{Z} , this notation has the advantage that evaluation at $u = 1$ (using the rule that $1^x = 1$ for all $x \in A$) recovers the cardinality of the multiset.

As an A -linear function, a Boltzmann weight can be conveniently expressed as an A -linear combination of characteristic functions

$$\chi_{x,y,z}(a,b,c) = \begin{cases} 1 & (a,b,c) = (x,y,z) \\ 0 & (a,b,c) \neq (x,y,z). \end{cases}$$

Example 5. Let X be the biquasile with operation matrix

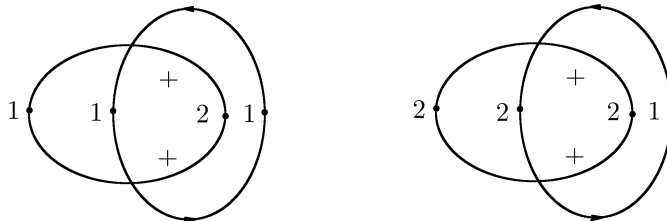
$$\left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{array} \right].$$

This biquasile can be written as $\mathbb{Z}_2 = \{1, 2\}$ with operations $x * y = x + y$ and $x \cdot y = x + y + 1$. Then our Python computations reveal 125 Boltzmann weights on X with values in \mathbb{Z}_5 , including for instance

$$\phi = 2\chi_{1,1,1} + 3\chi_{1,2,2} + 4\chi_{2,1,1} + 3\chi_{2,2,2}.$$

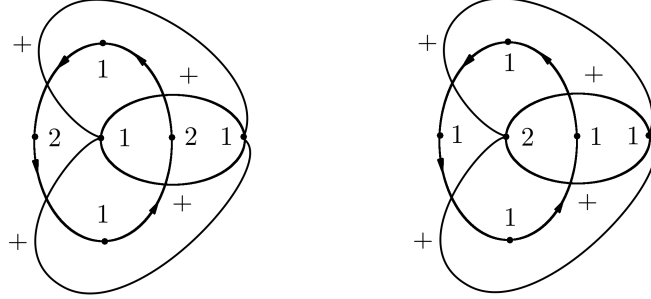
The Hopf link $L2a1$ and the $(4, 2)$ -torus link $L4a1$ have the same counting invariant value $\Phi_X^{\mathbb{Z}}(L2a1) = \Phi_X^{\mathbb{Z}}(L4a1) = 8$, distinguishing both from the unlink of two components which has counting invariant value $\Phi_X^{\mathbb{Z}}(U_2) = 4$. Let us use the Boltzmann enhancement to distinguish the two.

The Hopf link has eight biquasile colorings by X , including for instance



The coloring on the left has Boltzmann weight $2\phi(1, 1, 2) = 2(0) = 0$ while the coloring on the right has Boltzmann weight $\phi(2, 2, 2) + \phi(1, 2, 2) = 3 + 3 = 1$. Computing the other six Boltzmann weights, we obtain $\Phi_X^\phi(L) = 4 + 4u$.

The $(4, 2)$ -torus link has eight X -colorings including



The coloring on the left has Boltzmann weight $2\phi(1, 1, 1) + 2\phi(2, 1, 1) = 4 + 8 = 2$ while the coloring on the right has Boltzmann weight $4\phi(1, 2, 1) = 0$. Computing the other six Boltzmann weights, we obtain $\Phi_X^\phi(L) = 4 + 4u^2$, which distinguishes this link from the Hopf link. In particular, this example shows that the Boltzmann enhancement is a proper enhancement, i.e., a stronger invariant than the unenhanced biquasile counting invariant.

Example 6. Continuing with the biquasile X from example 5, we selected three Boltzmann weights with values in \mathbb{Z}_6 and computed $\Phi_X^\phi(L)$ for the list of prime links with up to seven crossings as listed in the Knot Atlas [1]; the results are collected in the table.

$$\begin{aligned}\phi_1 &= \chi_{1,1,1} + 5\chi_{1,2,2} + 3\chi_{2,1,1} + 5\chi_{2,2,2}, \\ \phi_2 &= \chi_{1,1,1} + \chi_{1,2,2} + 2\chi_{2,1,1} + 2\chi_{2,2,2} \quad \text{and} \\ \phi_3 &= 4\chi_{1,2,2} + 2\chi_{2,2,2}\end{aligned}$$

L	$L2a1$	$L4a1$	$L5a1$	$L6a1$	$L6a2$	$L6a3$	$L6a4$	$L6a5$	$L61$
$\Phi_X^{\phi_1}(L)$	$4 + 4u^4$	$4 + 4u^2$	8	$4 + 4u^2$	8	8	4	$4 + 12u^2$	$4 + 12u^2$
$\Phi_X^{\phi_2}(L)$	$4 + 4u^3$	8	8	8	$4 + 4u^3$	$4 + 4u^3$	4	16	16
$\Phi_X^{\phi_3}(L)$	8	8	8	8	8	8	4	16	16
L	$L7a1$	$L7a2$	$L7a3$	$L7a4$	$L7a5$	$L7a6$	$L7a7$	$L7n1$	$L7n2$
$\Phi_X^{\phi_1}(L)$	8	$4 + 4u^2$	8	8	$4 + 4u^4$	$4 + 4u^4$	$12 + 4u^2$	$4 + 4u^4$	8
$\Phi_X^{\phi_2}(L)$	8	8	8	8	$4 + 4u^3$	$4 + 4u^3$	16	8	8
$\Phi_X^{\phi_3}(L)$	8	8	8	8	8	8	16	8	8

We observe that the ϕ_3 weight yields just the biquasile counting invariant for the links in the table, while ϕ_1 and ϕ_2 both yield proper enhancements.

Proposition 3. Let $A = \mathbb{Z}_n$ and $X = \mathbb{Z}_n$ with a choice of $d, s, n \in X^\times$, making X a finite Alexander biquasile. Then for any $\gamma \in \mathbb{Z}_n$, the map $\phi : \mathbb{Z}_n[X]^3 \rightarrow \mathbb{Z}_n$ given by

$$\phi(x, y, z) = -\gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}dy + \gamma z$$

defines a Boltzmann weight which we call a linear Boltzmann weight.

Proof. Let $\phi(x, y, z) = -\gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}dy + \gamma z$. Recall that our biquasile operations are given by

$$x * y = -dsn^2x + ny \quad \text{and} \quad x \cdot y = dx + sy.$$

Then observing that

$$\begin{aligned}x \setminus^* y &= dsnx + n^{-1}y, \\x \setminus y &= -ds^{-1}x + s^{-1}y \text{ and} \\x/y &= d^{-1}x - d^{-1}sy,\end{aligned}$$

we compute

$$\begin{aligned}\phi(x, a, a \setminus (x \setminus^* x)) &= \phi(x, a, -ds^{-1}a + s^{-1}(dsnx + n^{-1}x)) \\&= -\gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}da + \gamma(-ds^{-1}a + s^{-1}(dsnx + n^{-1}x)) \\&= (-\gamma s^{-1}n^{-1} - \gamma dn + \gamma dn + \gamma s^{-1}n^{-1})x + (\gamma s^{-1}d - \gamma ds^{-1})a \\&= 0,\end{aligned}$$

$$\begin{aligned}\phi(x, (x \setminus^* x)/b, b) &= \phi(x, d^{-1}(dsnx + n^{-1}x) - d^{-1}sb, b) \\&= -\gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}d(d^{-1}(dsnx + n^{-1}x) - d^{-1}sb) + \gamma b \\&= (-\gamma s^{-1}n^{-1} - \gamma dn + \gamma s^{-1}dd^{-1}dsnx + \gamma s^{-1}dd^{-1}n^{-1})x + (-\gamma s^{-1}dd^{-1}s + \gamma)b \\&= (-\gamma s^{-1}n^{-1} - \gamma dn + \gamma dn + \gamma s^{-1}n^{-1})x + (-\gamma + \gamma)b \\&= 0\end{aligned}$$

so condition (i) is satisfied.

Checking condition (ii), we have on the left side

$$\begin{aligned}L &= \phi(x, a, b) + \phi(b, x * (a \cdot b), y) + \phi(x * (a \cdot b), a, b * ([x * (a \cdot b)] \cdot y)) \\&= -\gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}da + \gamma b - \gamma(s^{-1}n^{-1} + dn)b + \gamma s^{-1}d(-dsn^2x + nda + nsb) + \gamma y \\&\quad -\gamma(s^{-1}n^{-1} + dn)(-dsn^2x + nda + nsb) + \gamma s^{-1}da + \gamma(-dsn^2b + n[d(-dsn^2x + nda + nsb) + sy]) \\&= \gamma((-s^{-1}n^{-1} + dn) + s^{-1}d(-dsn^2) - (s^{-1}n^{-1} + dn)(-dsn^2) + nd(-dsn^2))x \\&\quad + \gamma(s^{-1}d + (s^{-1}d)nd - (s^{-1}n^{-1} + dn)nd + s^{-1}d + nd(dn))a \\&\quad + \gamma(1 + ns)y + \gamma(1 - (s^{-1}n^{-1} + dn) + s^{-1}dns - (s^{-1}n^{-1} + dn)ns - dsn^2 + ndns)b \\&= \gamma(-s^{-1}n^{-1} - dn - d^2n^2 + dn + d^2sn^3 - d^2sn^3)x + \gamma(s^{-1}d + s^{-1}nd^2 - s^{-1}d - d^2n^2 + s^{-1}d + d^2n^2)a \\&\quad + \gamma(1 + ns)y + \gamma(1 - s^{-1}n^{-1} - dn + dn - 1 - dsn^2 - dsn^2 + dsn^2)b \\&= \gamma(-s^{-1}n^{-1} - d^2n^2)x + \gamma(s^{-1}nd^2 + s^{-1}d)a + \gamma(1 + ns)y + \gamma(-s^{-1}n^{-1} - dsn^2)b\end{aligned}$$

while on the right side we have

$$\begin{aligned}R &= \phi(b, x, y) + \phi(x, a, b * (x \cdot y)) + \phi(b * (x \cdot y), x * (a \cdot [b * (x \cdot y)]), y) \\&= -\gamma(s^{-1}n^{-1} + dn)b + \gamma s^{-1}dx + \gamma y - \gamma(s^{-1}n^{-1} + dn)x + \gamma s^{-1}da + \gamma(-dsn^2b + ndx + nsy) \\&\quad -\gamma(s^{-1}n^{-1} + dn)(-dsn^2b + ndx + nsy) + \gamma s^{-1}d(-dsn^2x + nda + ns(-dsn^2b + ndx + nsy)) + \gamma y \\&= \gamma(s^{-1}d - (s^{-1}n^{-1} + dn) + nd - (s^{-1}n^{-1} + dn)nd + s^{-1}d(-dsn^2 + nsnd))x \\&\quad + \gamma(s^{-1}d + s^{-1}dnd)a + \gamma(1 + ns - (s^{-1}n^{-1} + dn)ns + s^{-1}d(nsns) + 1)y \\&\quad + \gamma(-(s^{-1}n^{-1} + dn) - dsn^2 - (s^{-1}n^{-1} + dn)(-dsn^2) + s^{-1}dns(-dsn^2))b \\&= \gamma(s^{-1}d - s^{-1}n^{-1} - dn + nd - s^{-1}d - d^2n^2 - d^2n^2 + d^2n^2)x \\&\quad + \gamma(s^{-1}d + s^{-1}nd^2)a + \gamma(1 + ns - 1 - dsn^2 + dsn^2 + 1)y \\&\quad + \gamma(-s^{-1}n^{-1} - dn - dsn^2 + dn + d^2sn^3 - d^2sn^3)b \\&= \gamma(-s^{-1}n^{-1} - d^2n^2)x + \gamma(s^{-1}d + s^{-1}nd^2)a + \gamma(1 + ns)y + \gamma(-s^{-1}n^{-1} - dsn^2)b\end{aligned}$$

as required. □

Example 7. The Alexander biquasile \mathbb{Z}_3 with $d = s = 2$ and $n = 1$ has operation matrix

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 & 3 \end{array} \right].$$

Then up to scalar multiplication, it has one linear Boltzmann weight $\phi : \mathbb{Z}_3[X^3] \rightarrow \mathbb{Z}_3$ given by

$$\begin{aligned} \phi = & 2\chi_{1,1,1} + \chi_{1,1,2} + \chi_{1,2,1} + 2\chi_{1,2,3} + 2\chi_{1,3,2} + \chi_{1,3,3} \\ & + 2\chi_{2,1,2} + \chi_{2,3,1} + 2\chi_{2,2,1} + \chi_{2,2,2} + \chi_{2,3,1} + 2\chi_{2,3,3} \\ & + \chi_{3,1,1} + 2\chi_{3,1,3} + 2\chi_{3,2,2} + \chi_{3,2,3} + 2\chi_{3,3,1} + \chi_{3,3,2}. \end{aligned}$$

Our Python computations say that this Boltzmann weight defines a trivial enhancement (i.e., just the counting invariant) for all prime knots with up to 8 crossings and all prime links with up to 7 crossings. However, this is not the only trivial case. In fact,

Conjecture 1. *The linear Boltzmann weights as defined by Proposition 3 is always a trivial enhancement.*

Our Python computations reveal that all Alexander biquasiles over \mathbb{Z}_n where $0 < n \leq 7$ only have trivial linear Boltzmann enhancements on prime classical knots with up to eight crossings and prime classical links with up to seven crossings.

4 Questions

We end with a few collected questions for future research.

- In several recent papers such as [8, 9], Niebrzydowski and collaborators have studied region colorings of knots and link by structures called *knot-theoretic ternary quasigroups* and have introduced a homology theory for these structures. A biquasile determines a knot-theoretic ternary quasigroup by

$$abcT = b * (a \cdot c)$$

and our Boltzmann weights correspond to these cocycles. What is the exact relationship between these structures?

- Given two distinct knots or links, is there always a pair of biquasile and Boltzmann weight such that the enhanced invariant distinguishes the pair?
- Is Conjecture 1 true?
- What are the conditions for polynomial Boltzmann weights of higher degree for Alexander biquasiles analogous to the linear Boltzmann weights in Proposition 3?

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