Solving Quartic Equations

The Abel-Ruffini theorem states that no general solutions exist for polynomials of degree five or higher.

This requires a substantial (but beautiful) amount of algebra to prove. Fortunately, however, it is much easier to prove that general solutions exist for the polynomials of degree four or lower, which I outline here!

Theorem 1 (Solvability of the Quartics)**.** *For any equation $x^4 + bx^3 + cx^2 + dx + e \in \mathbb{C}[x]$, we can definitively find their roots.*

This means to say, that for any quartic equation, that we can find its roots like we can for the quadratic $x^2 + ax + b$ with the so-called "quadratic formula" $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Here, we prove that we can find such a formula for quartic as well.

We first need to prove that a formula for the cubic also exists.

Theorem 2 (Solvability of the Cubics)**.** *For any equation $x^3 + ax^2 + bx + c = 0 \in \mathbb{C}[x]$, we can definitively find their roots.*

Proof. For any $x^3 + ax^2 + bx + c = 0$, substitute x with $y = x - \frac{b}{3}$ and obtain

$$y^3 + py + q = 0$$

where $p = \frac{a^2}{3} - \frac{2a}{3} + b$ and $q = \frac{-a^3}{27} + \frac{a^2}{9} - \frac{ab}{3} + c$.

Now, we make another substitution, $y = z - \frac{p}{3z}$ to obtain that

$$z^3 - \frac{p^3}{27z^3} + q = 0$$

and equivalently

$$(z^3)^2 + qz^3 - \frac{p^3}{27z^3} = 0.$$

Using the quadratic formula, we can find the roots of z^3 as

$$\frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} \tag{1}$$

Let these roots be A and B. These A and B are conjugate imaginary terms, and their product yields $\frac{-p^3}{27}$ and we find that $\sqrt[3]{AB} = \frac{-p}{3}$.

Certainly, for these $A, B = z^3$, we know that $\sqrt[3]{A}$ and $\sqrt[3]{B}$ are roots of z. But what else are the roots of z? We know that Euler's equation gives us e^{ix} =$\cos(x) + i\sin(x)$, and plugging into $2k\pi$ for x for some $k \in \mathbb{Q}$, we have the infamous polar coordinate of 1 as

$$e^{2k\pi i} = 1.$$

We can express this equation as

$$(e^{\frac{2}{3}k\pi i})^3 = 1,$$

and solving for these values will give use the numbers that cube to 1 (the so-called 3rd roots of unity).

Plugging in 0 gives us $e^0 = 1$. Plugging in 1 into k, we get that

$$e^{\frac{2}{3}\pi i} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1 + i\sqrt{3}}{2}$$

Let this be ω. Computationally, we can check that indeed $\omega^3 = 1$. Similarly, plugging in 2 into k, we get that $\omega^2 = \frac{-1-i\sqrt{3}}{2}$ is also a cube root of 1.

Therefore, since we have A and B as solutions for z^3, we can expand our possible for solutions for z to the following:

$$\sqrt[3]{A}, \sqrt[3]{B}, \omega\sqrt[3]{A}, \omega\sqrt[3]{B}, \omega^2\sqrt[3]{A}, \omega^2\sqrt[3]{B}.$$

Now, we plug these solutions to z back into $y = z - \frac{p}{3z}$. Let us first try $z = \sqrt[3]{A}$. Using the fact that we found that $\sqrt[3]{AB} = \frac{-p}{3}$, we can get that

$$y = \sqrt[3]{A} - \frac{p}{3\sqrt[3]{A}} = \sqrt[3]{A} - \frac{p\sqrt[3]{B}}{3\sqrt[3]{A}\sqrt[3]{B}} = \sqrt[3]{A} - \frac{p\sqrt[3]{B}}{3\frac{-p}{3}} = \sqrt[3]{A} + \sqrt[3]{B}.$$

Plugging in $z = \sqrt[3]{B}$ gives us the same solution. What about then $z = \omega\sqrt[3]{A}$?

$$y = \omega\sqrt[3]{A} - \frac{p}{3\omega\sqrt[3]{A}} = \omega\sqrt[3]{A} - \frac{p\omega^2\sqrt[3]{B}}{\omega\sqrt[3]{A}\omega^2\sqrt[3]{B}} = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}.$$

Plugging in $z = \omega^2\sqrt[3]{B}$ gives the solution, and similarly, plugging in $z = \omega^2\sqrt[3]{A}$ gives us our final answer that

$$y = \omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B}.$$

Putting these all together with Equation (1), we get that

$$y = \omega^i \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} = \omega^2 \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

for $i = 0, 1, 2$. Since $p = \frac{a^2}{3} - \frac{2a}{3} + b$ and $q = \frac{-a^3}{27} + \frac{a^2}{9} - \frac{ab}{3} + c$, and all values of a, b, c are given, we see that we can definitively find the roots of the cubic.

□

Now we can tackle the solvability of the quartic.

Theorem 1 (Solvability of the Quartics)**.** *For any equation* $x^4 + bx^3 + cx^2 + dx + e \in \mathbb{C}[x]$*, we can definitively find their roots.*

Proof. For a given $x^4 + bx^3 + cx^2 + dx + e = 0$, let us substitute x with the variable $y = x + \frac{a}{4}$. Then we can rewrite our equation as

$$(y - \frac{a}{4})^4 + a(y - \frac{a}{4})^3 + b(y - \frac{a}{4})^2 + c(y - \frac{a}{4}) + d = 0$$

$$y^4 + (-\frac{3a}{8} + b)y^2 + (\frac{a}{8} - \frac{ab}{2} + c)y + (\frac{3a^4}{256} + \frac{a^2}{16} - \frac{ac}{4} + d) = 0$$

Let $p = -\frac{3a}{8} + b$ and $q = \frac{a}{8} - \frac{ab}{2} + c$ and $r = \frac{3a^4}{256} + \frac{a^2}{16} - \frac{ac}{4} + d$, and put our equation in the form of

$$y^4 + py^2 + qy + r = 0$$

and equivalently

$$y^4 = -(py^2 + qy + r).$$

Now, we add $y^2 z + \frac{1}{4}z^2$ to both sides to obtain

$$y^4 + y^2 z + \frac{1}{4}z^2 = -py^2 + y^2 z - qy - r + \frac{1}{4}z^2$$

and therefore

$$(y^2 + \frac{1}{2}z)^2 = (z - p)y^2 - qy + (\frac{1}{4}z^2 - r).$$

This implies that the right hand side is a square of some term. However, the right hand side has degree two, so it must a square of a linear term, and therefore

$$(z-p)y^2 - qy + (\frac{1}{4}z^2 - r) = (my+k)^2 \tag{2}$$

for some m and k. This implies that this quadratic equation has only one root, and that only occurs when its determinant is zero. In this case, the determinant is $\sqrt{q^2 - 4(z-p)(\frac{1}{4}z^2 - r)}$, so we have that

$$q^2 - 4(z-p)(\frac{1}{4}z^2 - r) = 0$$

and

$$z^3 - pz^2 - 4rz + (4pr - q^2) = 0.$$

Here, by Theorem 2, we know that we can find values of z. Plugging the values of z back into Equation (2), we can find values for m and k. Then we can plug these values of z, m and k for

$$(y^2 + \frac{1}{2}z)^2 = (my+k)^2$$

or rather,

$$y^2 + \frac{1}{2}z = my + k$$

Which we know how to solve.

□