Boolean Algebras are Posets

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Definition 1. *A Boolean Algebra is a* partially ordered set (poset) B *equipped with distinguished elements* 0, 1, *binary operations* $a \vee b$ *of "join", and* $a \wedge b$ *of "meet", and a unary operation* $\neg b$ *of "complementation". For any arbitrary* $a, b, c \in B$, *the operations are required satisfy the following conditions:*

$$0 \leq a$$
$$a \leq 1$$
$$a \leq c \textit{ and } b \leq c \textit{ iff } a \vee b \leq c$$
$$c \leq a \textit{ and } c \leq b \textit{ iff } c \leq a \wedge b$$
$$a \leq \neg b \textit{ iff } a \wedge b = 0$$
$$\neg\neg a = a$$

Example 1 (Powersets as Boolean Algebras)**.** *Let* $P(X)$ *be the powerset of all subsets* $A \subseteq X$ *of a set* X. *Then, taking the inclusion operator* $A \subseteq B$ *as the ordering operator, the empty set* $\varnothing$ *as* 0, *the whole set* X *as* 1, *union and intersection of subsets as join and meet, and the relative complement* $X - A$ *as* $\neg A$, $P(X)$ *is a Boolean Algebra.*

Example 2 (Standard Boolean Algebra)**.** *Given* $\top$ *(true) as* 1, $\bot$ *(false) as* 0, *and the familiar boolean disjunction, conjunction, negation as join, meet, and complementation, the set* $\{\top, \bot\}$ *is a Boolean Algebra.*